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Phase ordering from off-critical quenches and the measurement of the dynamic exponent λ

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Abstract. The ordering dynamics of a system with non-conserved order parameter is considered following a quench into the ordered phase from high temperature. The correlation of the order-parameter field with its initial condition (or, more generally, the correlation between the fields at different times) involves the dynamic exponent λ, a non-trivial exponent associated with the $T = 0$ fixed point that drives phase ordering. It is shown that λ can be determined from the growth of the mean order parameter $m(t)$ in a system which has a small but non-zero mean order parameter m_0 at $t = 0$, since $m(t) \sim m_0 L(t)^\lambda$ where $L(t) \sim t^{1/2}$ is the characteristic length scale at time t. The role of a weak external field h is also considered: for $h \neq 0$ the growth of $m(t)$ involves two power-law terms, $m(t) \sim h(a\,t^x + b\,t^{\lambda/2})$ (for $m_0 = 0$), with $x = \frac{1}{2}$ or 1 for a scalar or vector order parameter, respectively. The results are illustrated by the exact solution of the O(n) model at large n for general values of m_0 and h.

1. Introduction

Systems quenched from their high-temperature phase to the low-temperature (ordered) phase exhibit complex nonlinear phenomena. The dynamics of the resulting domain growth has attracted much interest [1], because it obeys *dynamic scaling* at late times, i.e. spatial correlations are time independent when lengths are measured in units of the characteristic scale ('domain size') $L(t)$, where $L(t) \sim t^{1/2}$ for a non-conserved order parameter [1, 2].

Much recent work has been devoted to the study of 'two-time' correlations, i.e. the correlation between the order-parameter fields at two different times, which require a new, non-trivial exponent for their description [3, 4]. For example, Newman and Bray [4] have used a $1/n$ expansion for an n-component vector order parameter to calculate the dynamic correlation function

$$C(\boldsymbol{k}, t, t') = [S_i(\boldsymbol{k}, t) S_i(-\boldsymbol{k}, t')] \tag{1}$$

where $S_i(\boldsymbol{k}, t)$ is a Fourier component of the 'spin field' (we use magnetic language throughout) at time t and i indicates a cartesian component in spin space. The brackets in (1) denote an average over an ensemble of initial conditions. $C(\boldsymbol{k}, t, t')$ was found to depend on the characteristic length scale $L(t) \sim t^{1/2}$ and a new exponent λ, which depends on both the spatial dimension d and the spin dimension n. For $t \gg t'$,

$$C(\boldsymbol{k}, t, t') = L(t')^d \left(\frac{L(t)}{L(t')}\right)^\lambda g(|\boldsymbol{k}| L(t)) \tag{2}$$

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where $g(x)$ is a scaling function satisfying $g(0)$ =constant.

The exponent λ has been calculated to first order in $1/n$ [4], and computer simulations [5, 6] are in good agreement with the $1/n$ results. So far, however, no one has suggested a method to measure λ experimentally. The main difficulty with testing (2) directly is that it is difficult to determine experimentally correlations between fields at different times. The purpose of this paper is to show that this problem can be overcome by introducing a small initial magnetization m_0 (we shall use *magnetization* for the mean of the order parameter), i.e. to perform an 'off-critical' quench. It will turn out that in the scaling regime $m(t)$ is expected to grow like

$$m(t) \sim m_0\, L(t)^{\lambda}. \tag{3}$$

This result, which relates $m(t)$ and $C(\boldsymbol{k}, t, 0)$, will be derived via a graph expansion. For completeness, we will also consider the effect of an external field h, which is more complicated and leads to a growth law for $m(t)$ containing two competing power laws.

A first insight into the problem will be achieved in section 2 by considering the *fixed-length spin* model in the limit $n \to \infty$. Both $m(t)$ and $C(\boldsymbol{k}, t, t')$ (and therefore λ) can be calculated explicitly in this case. We will also see that the role of h as a driving force for the growth of $m(t)$ depends on the spatial dimension of the system: for h =constant, the early induced magnetization is the dominating growth mechanism if $\lambda > 2$, whereas h remains important for all t when $\lambda < 2$. In the appendix we show that the differential equation governing phase ordering in a magnetic field is related to a standard differential equation, Abel's equation.

In section 3 a diagrammatic expansion of the time-dependent Landau–Ginzburg equation, which is the soft-spin equivalent of the hard-spin model discussed in section 2, is used to derive equation (3) more generally: no large-n approximation is required, and equation (3) is shown to be correct in general to leading order in m_0. We also show that the effect of a weak external field can be obtained by rather general arguments. For a vector order parameter we find (for $m_0 = 0$) $m(t) \sim h(at + bt^{\lambda/2})$ to leading order in h, in agreement with the explicit result obtained for the large-n limit. For a scalar order parameter $m(t) \sim h(at^{1/2} + bt^{\lambda/2})$ is predicted.

Section 4 concludes with a summary of the results.

2. The large-n limit

2.1. The model

We consider a system of classical fixed-length spins $\boldsymbol{S}(\boldsymbol{x}, t)$, where $\boldsymbol{S}$ has n components and $\boldsymbol{S}^2 = 1$. The Hamiltonian with an external field $\boldsymbol{h}(\boldsymbol{x}, t)$ is given by

$$\mathcal{H} = -\tfrac{1}{2}\sum_{\boldsymbol{x},\mu} \boldsymbol{S}(\boldsymbol{x}) \cdot \boldsymbol{S}(\boldsymbol{x}+\mu) - \sum_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}) \cdot \boldsymbol{S}(\boldsymbol{x}) \tag{4}$$

where μ runs through the nearest neighbours of $\boldsymbol{x}$; the exchange energy J has been set equal to unity.

In the same spirit as Newman *et al* [5], we can derive the equation of motion for a non-conserved order parameter:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{S}(\boldsymbol{x},t)}{\mathrm{d}t} &= \sum_{\mu} \boldsymbol{S}(\boldsymbol{x}+\mu,t) + \boldsymbol{h}(\boldsymbol{x},t)\\ &\quad - \left\{ \left(\sum_{\mu} \boldsymbol{S}(\boldsymbol{x}+\mu,t) + \boldsymbol{h}(\boldsymbol{x},t) \right) \cdot \boldsymbol{S}(\boldsymbol{x},t) \right\} \boldsymbol{S}(\boldsymbol{x},t).\end{aligned} \tag{5}$$

The kinetic coefficient has been scaled to 1 by choice of units of time. The first two terms on the right-hand side are obtained by differentiating the Hamiltonian with respect to $\boldsymbol{S}(\boldsymbol{x})$. As we are dealing with spins of fixed length, the rate of change of a spin has to be perpendicular to it, which is achieved by subtracting the parallel component of the force, which is just the third term. We also choose the temperature $T = 0$ for convenience because the phase-ordering process is governed by the stable zero-temperature fixed point [4, 7]. The only effect of a non-zero temperature is the modification of amplitudes [4, 7], which is the same as saying the temperature is an irrelevant variable in the renormalization group sense. Of course, T has to be below its critical value.

In the continuum limit equation (5) (with the lattice spacing taken as unity) reads

$$\frac{\partial \boldsymbol{S}(\boldsymbol{x},t)}{\partial t} = \nabla^2 \boldsymbol{S}(\boldsymbol{x},t) + \boldsymbol{h}(\boldsymbol{x},t) - \{(\nabla^2 \boldsymbol{S}(\boldsymbol{x},t) + \boldsymbol{h}(\boldsymbol{x},t)) \cdot \boldsymbol{S}(\boldsymbol{x},t)\}\boldsymbol{S}(\boldsymbol{x},t) \tag{6}$$

where we have neglected higher order spatial derivatives.

For $n \to \infty$ it is convenient to rescale $\boldsymbol{S} \to \boldsymbol{S}/\sqrt{n}$, $\boldsymbol{h} \to \boldsymbol{h}/\sqrt{n}$. The fixed length property implies the equation $\boldsymbol{S} \cdot \nabla^2 \boldsymbol{S} = -(\nabla \boldsymbol{S})^2$, which we use to write (6) as

$$\frac{\partial \boldsymbol{S}(\boldsymbol{x},t)}{\partial t} = \nabla^2 \boldsymbol{S}(\boldsymbol{x},t) + \boldsymbol{h}(\boldsymbol{x},t) + \frac{1}{n}\{(\nabla \boldsymbol{S}(\boldsymbol{x},t))^2 - \boldsymbol{h}(\boldsymbol{x},t) \cdot \boldsymbol{S}(\boldsymbol{x},t)\}\boldsymbol{S}(\boldsymbol{x},t). \tag{7}$$

We now choose $\boldsymbol{h}(\boldsymbol{x},t)$ to be spatially uniform and pointing into the (1, 1, ..., 1) direction, i.e.

$$h_i(\boldsymbol{x},t) = h(t) \qquad \text{for all } i. \tag{8}$$

The latter choice is for computational convenience.

Let us further prepare the system at $t = 0$ in such a way that there is an average magnetization $\sqrt{n}m_0$ in the $(1, 1, \ldots, 1)$ direction, but no additional correlations,

$$[S_i(\boldsymbol{x},0)] = m_0 \tag{9}$$

$$[S_i(\boldsymbol{x},0)S_j(\boldsymbol{y},0)] = m_0^2 + (1 - m_0^2)\,\delta_{i,j}\,\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{10}$$

The square brackets denote the average over an ensemble of initial conditions. The correlator has been chosen to ensure that $[\boldsymbol{S}(\boldsymbol{x},0)^2] = \boldsymbol{S}(\boldsymbol{x},0)^2 = n$. We do not need to specify higher cumulants because they will not enter the calculation; they are irrelevant. Equation (7) and the initial conditions model a system which is initially at temperature $T = \infty$ with a bias towards the (1, 1, ..., 1) direction and is quenched to $T = 0$ at $t = 0$.

In order to solve (7) analytically, we now take the number of components to be infinitely large ('spherical limit'). For $n \to \infty$, (7) is equivalent to

$$\frac{\partial \boldsymbol{S}(\boldsymbol{x},t)}{\partial t} = \nabla^2 \boldsymbol{S}(\boldsymbol{x},t) + \boldsymbol{h}(\boldsymbol{x},t) + [(\nabla S_l(\boldsymbol{x},t))^2 - h(\boldsymbol{x},t)S_l(\boldsymbol{x},t)]\boldsymbol{S}(\boldsymbol{x},t) \tag{11}$$

where S_l is an arbitrary component of $\boldsymbol{S}$. There is no summation over components left because the average $[\ldots]$ is the same for each component, and the summation has been carried out, cancelling the factor $1/n$ in (7).

Equation (11) is conveniently solved in k-space, where the initial conditions read

$$[S_i(\boldsymbol{k},0)] = \sqrt{N} m_0 \delta_{\boldsymbol{k},0} \tag{12}$$

$$[S_i(\boldsymbol{k},0)S_j(-\boldsymbol{k},0)] = N m_0^2 \delta_{\boldsymbol{k},0} + \delta_{i,j}\,(1-m_0^2) \tag{13}$$

where N is the number of spins in the system. Equation (11) can be rewritten

$$\frac{\mathrm{d}S_i(\boldsymbol{k},t)}{\mathrm{d}t} = \left(-k^2 + \frac{1}{N}\sum_{p} p^2[S_l(\boldsymbol{p},t)S_l(-\boldsymbol{p},t)] - h(t)m(t)\right) S_i(\boldsymbol{k},t) + \sqrt{N} h \delta_{\boldsymbol{k},0} \tag{14}$$

where $m(t)$ is the average magnetization per spin for a given component (i.e. the total magnetization per spin is $\sqrt{n}\, m(t)$):

$$m(t) = [S_l(\boldsymbol{x},t)] = \frac{1}{\sqrt{N}}\sum_{p}[S_l(\boldsymbol{p},t)] = \frac{1}{\sqrt{N}}[S_l(0,t)] \tag{15}$$

since $[S_l(\boldsymbol{p},t)] = 0$ for $\boldsymbol{p} \neq 0$. In all cases momentum sums are over the first Brillouin zone.

2.2. Solution of the differential equation

The differential equation (14) can be solved explicitly for $h(t) = 0$, and solved numerically to any desired accuracy for arbitrary h.

Integrating (14) gives

$$S_i(\boldsymbol{k},t) = \begin{cases} S_i(\boldsymbol{k},0)\mathrm{e}^{-k^2 t}\mathrm{e}^{Q(t)} & \text{for } k \neq 0 \\ \left(S_i(0,0) + \int_0^t \mathrm{d}t' \sqrt{N} h(t')\mathrm{e}^{-Q(t')}\right)\mathrm{e}^{Q(t)} & \text{for } k = 0 \end{cases} \tag{16}$$

where $Q(t)$ is defined as

$$Q(t) = \int_0^t \mathrm{d}t' \left(\frac{1}{N}\sum_{p} p^2[S_l(\boldsymbol{p},t')S_l(-\boldsymbol{p},t')] - h(t')m(t')\right). \tag{17}$$

From (16) and (17) it is easy to derive equations for $m(t)$ and $\dot{Q}(t)$:

$$m(t) = \left(m_0 + \int_0^t \mathrm{d}t' h(t')\mathrm{e}^{-Q(t')}\right)\mathrm{e}^{Q(t)} \tag{18}$$

$$\dot{Q}(t) = (1-m_0^2)\frac{1}{N}\sum_{p} p^2 \mathrm{e}^{-2p^2 t}\mathrm{e}^{2Q(t)} - h(t)\left(m_0 + \int_0^t \mathrm{d}t' h(t')\mathrm{e}^{-Q(t')}\right)\mathrm{e}^{Q(t)}. \tag{19}$$

Equation (19) can be integrated, with initial condition $Q(0) = 0$, to obtain

$$\mathrm{e}^{-2Q(t)} = (1-m_0^2)\frac{1}{N}\sum_{p} \mathrm{e}^{-2p^2 t} + \left(m_0 + \int_0^t \mathrm{d}t' h(t')\mathrm{e}^{-Q(t')}\right)^2. \tag{20}$$

Note that in this equation $Q(t)$ enters only through the function $\mathrm{e}^{Q(t)}$. To make further progress, we treat the cases $h(t) = 0$ and $m_0 = 0$ separately.

2.2.1. $h(t) = 0$. For vanishing external field h, (20) simplifies to

$$e^{Q(t)} = \left((1 - m_0^2)\frac{1}{N}\sum_p e^{-2p^2 t} + m_0^2\right)^{-1/2} . \tag{21}$$

Using (18) (with $h = 0$), $m(t)$ is given by

$$m(t) = m_0 \left((1 - m_0^2)\frac{1}{N}\sum_p e^{-2p^2 t} + m_0^2\right)^{-1/2} . \tag{22}$$

For $t \gg 1$ the exponential function is negligible for wave vectors outside the first Brillouin zone and the integration can be extended to infinity with negligible error. We can therefore approximate

$$\frac{1}{N}\sum_p e^{-2p^2 t} \simeq \left(\frac{t_0}{t}\right)^{d/2} \tag{23}$$

with $t_0 = (8\pi)^{-1}$.

Equation (22) shows that, ultimately, $m(t)$ saturates to unity, as expected. The scaling regime, which is our main interest [8], is for times $\left(\frac{t_0}{t}\right)^{d/2} \gg m_0^2$, where $m(t)$ grows with a power law:

$$m(t) = m_0 \left(\frac{t}{t_0}\right)^{d/4} . \tag{24}$$

Clearly a wide scaling regime requires $m_0 \ll 1$. One can also calculate $C(\boldsymbol{k}, t)$, the correlation function between 0 and t:

$$C(\boldsymbol{k}, t) = [S_i(\boldsymbol{k}, t) S_i(-\boldsymbol{k}, 0)]. \tag{25}$$

It is given by

$$C(\boldsymbol{k}, t) = \begin{cases} e^{-\boldsymbol{k}^2 t} e^{Q(t)} = (t/t_0)^{d/4} e^{-\boldsymbol{k}^2 t} & \text{for } k \neq 0 \\ N m_0 m(t) & \text{for } k = 0 \end{cases} \tag{26}$$

in the scaling regime.

The exponent λ can be read off from (26): $\lambda = d/2$, in agreement with previous calculations [4, 5, 9] for $n = \infty$. Note that (24) can be rewritten as $m(t) = m_0 (t/t_0)^{\lambda/2}$, which is a first indication that λ can be determined by measuring $m(t)$. We show below that this result is valid generally, not just in the large-n limit. This result assumes, of course, that the total magnetization is far from its saturation value, i.e. the system is in its scaling regime. Eventually the system leaves the scaling regime and the magnetization saturates.

2.2.2. $h(t) \neq 0$. We now turn our attention to the case of non-zero external field. Equation (20) can be made more transparent by substituting

$$f(t) = \int_0^t \mathrm{d}t' h(t') \mathrm{e}^{-Q(t')} \tag{27}$$

with the initial condition $f(0) = 0$ and $\dot{f}(0) = h(0)$. From the definition of f, $\dot{f}(t)$ has to have the same sign as $h(t)$. We obtain from (20)

$$\dot{f}(t) = h(t)\left((1-m_0^2)\frac{1}{N}\sum_p \mathrm{e}^{-2p^2t} + (m_0 + f(t))^2\right)^{1/2}. \tag{28}$$

In the appendix we transform this equation into the well-known Abel differential equation, but the approximation required to solve it in the scaling regime can be seen on physical grounds by noting from (18) and (27) that

$$m(t) = h(t)\frac{m_0 + f(t)}{\dot{f}(t)}. \tag{29}$$

This is a useful identity because it implies that in the scaling regime, where $m(t)^2 \ll 1$, the final term in the large brackets in (28) can be dropped, to give the simple equation

$$\dot{f}(t) = h(t)\left((1-m_0^2)\frac{1}{N}\sum_p \mathrm{e}^{-2p^2t}\right)^{1/2}. \tag{30}$$

Using (29) gives the time dependence of m for $m(t) \ll 1$:

$$m(t) = \left(\frac{t}{t_0}\right)^{d/4} m_0 + \int_0^t \mathrm{d}t' h(t') \left(\frac{t}{t'}\right)^{d/4}. \tag{31}$$

Again, the momentum sum has been replaced by (23) to display the essence of the solution more clearly.

Note that the two contributions to $m(t)$ have different physical origins: the first comes from the initial magnetization, and is the previously calculated zero-field result; the second is due explicitly to the external field. Which term dominates (for $t \gg t_0$ but $m(t) \ll 1$) depends on the spatial dimension d and the explicit form of $h(t)$.

As a nice application, note that the external field enables us to avoid the initial transient behaviour at early times ($t < t_0$), which is caused by the cut-off of the momentum sum, i.e. the approximation (23). This is done in the following way. Take $h(t)$ non-zero only in a time interval $[t_1, t_2]$, $t_1 \gg t_0$. Then from (31) $m(t)$ has the very simple form

$$m(t) = \begin{cases} 0 & \text{for } t < t_1 \\ m(t_2)(t/t_2)^{d/4} & \text{for } t > t_2. \end{cases} \tag{32}$$

This result shows that a magnetization induced at later times grows with the same power law as the magnetization evolving from an initial value, but with a different timescale and without the initial cross-over, because the system is already in its scaling regime when the magnetization starts to grow.

Setting $h = h_0$ =constant provides an insight into the nature of the interplay between growth of magnetization 'by itself' and growth of magnetization driven by the external field. Setting $m_0 = 0$ in (31):

$$m(t) = h_0 \int_{t_0}^{t} \mathrm{d}t' \left(\frac{t}{t'}\right)^{d/4} \tag{33}$$

where the lower cut-off t_0 has been introduced as a simple way of taking into account the breakdown of (23) at short times. For $d > 4$, the integral is dominated by short times and

$$m(t) \propto \left(\frac{t}{t_0}\right)^{d/4} \left[1 - \left(\frac{t}{t_0}\right)^{1-d/4}\right] . \tag{34}$$

The second factor goes to one for $t/t_0 \gg 1$. This means (compare with (24)), that once a magnetization is induced, it grows basically as if there were no external field, the growth 'by itself' being the important process. For $d < 4$ the situation is different: the integral is dominated by late times, which means the continued influence of h is the relevant driving force. In this case

$$m(t) \propto t \left[1 - \left(\frac{t}{t_0}\right)^{d/4-1}\right] \tag{35}$$

is proportional to t for $t/t_0 \gg 1$.

Figure 1 shows the numerical solution and the analytical solution (broken lines), for dimensions $1 \leqslant d \leqslant 5$, with $m_0 = 0$ and $h = 0.01$. The analytically calculated curves include the corrections to the leading behaviour, i.e. those terms in brackets in the above two equations. The analytical results, derived in the scaling regime, describe the full solution (which includes saturation effects) very well when $m < 0.25$. In the next section, equations (34) and (35) will be generalized to any value of n by using rather simple physical arguments.

3. Finite n and the Landau–Ginzburg model

3.1. The model

For the $n = \infty$ limit we have shown that the magnetization $m(t)$ is closely related to the correlation function with the initial condition $C(\boldsymbol{k}, t)$. We now consider the number of components n to be finite and show that the basic result, i.e. the relation between $m(t)$ and $C(\boldsymbol{k}, t)$, is unchanged. For this section it is convenient to use a 'soft-spin' model, instead of the 'hard-spin' (or 'fixed-length spin') model used in the previous section.

The dynamics of the soft-spin model are described by the time-dependent Landau–Ginzburg equation:

$$\frac{\partial \phi_i(\boldsymbol{k}, t)}{\partial t} = (r - k^2)\phi_i(\boldsymbol{k}, t) - \frac{u}{n}\frac{1}{N}\sum_{\boldsymbol{p},\boldsymbol{q};j} \phi_j(\boldsymbol{p}, t)\phi_j(\boldsymbol{q}, t)\phi_i(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q}, t) + h_i(\boldsymbol{k}, t). \tag{36}$$

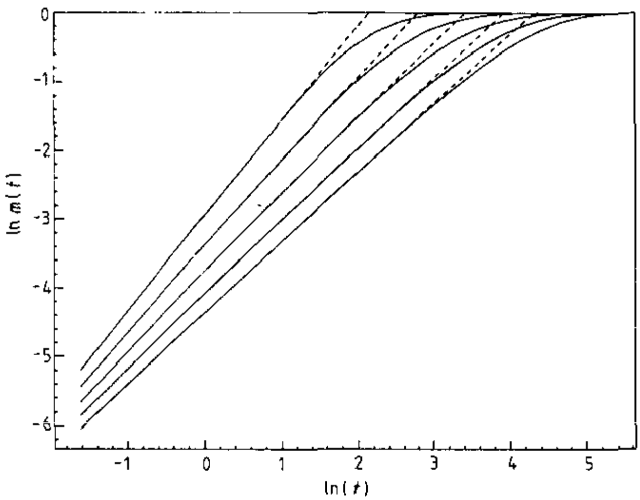

Figure 1. Time dependence of the magnetization $m(t)$, plotted as $\ln m$ against $\ln t$, for the O(n) model with $n = \infty$. The continuous curves were obtained by solving (28) numerically, with $h = 0.01$ and $m_0 = 0$, and substituting the solution into (29). The five curves represent (top to bottom) spatial dimensions $d = 5, 4, 3, 2, 1$. The broken curves are obtained from the corresponding solutions of the simpler equation (30), valid for $m(t) \ll 1$.

For $r/u = 1$ and $r \to \infty$ the length of the order parameter is forced asymptotically to be $|\phi|^2 = n$, and the hard-spin case is recovered. We expect both cases to be in the same universality class: for $n \to \infty$ it can be explicitly shown how the equation of motion (36) leads to the fixed-length result.

The reason we choose the soft-spin model is that, for finite n, Gaussian initial conditions do not satisfy the fixed-length condition exactly, which would introduce an additional technical problem. For the soft spins we are free to assume the simple Gaussian form:

$$[\phi_i(\boldsymbol{k}, 0)] = \sqrt{N} m_0 \delta_{\boldsymbol{k},0} \tag{37}$$

$$[\phi_i(\boldsymbol{k}, 0)\phi_j(-\boldsymbol{k}, 0)] = \Delta\delta_{i,j} + N m_0^2 \delta_{\boldsymbol{k},0}. \tag{38}$$

Additionally we want the external field to be

$$h_i(\boldsymbol{k}, t) = \sqrt{N} h(t) \delta_{\boldsymbol{k},0} \tag{39}$$

where N is the number of spins in the system. The initial preparation of the system is therefore basically the same as the one we have chosen for the hard-spin case.

3.2. Diagrammatic analysis

Equation (36) can be formally integrated to the following equation:

$$\phi_i(\boldsymbol{k}, t) = \mathrm{e}^{(r-\boldsymbol{k}^2)t}\phi_i(\boldsymbol{k}, t) + \int_0^t \mathrm{d}t' \exp[(r - \boldsymbol{k}^2)(t - t')] h_i(\boldsymbol{k}, t')$$

$$- \frac{u}{n}\int_0^t \mathrm{d}t' \exp[(r - \boldsymbol{k}^2)(t - t')] \frac{1}{N} \sum_{\boldsymbol{p},\boldsymbol{q};j} \phi_j(\boldsymbol{p}, t')\phi_j(\boldsymbol{q}, t')\phi_i(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q}, t') \tag{40}$$

This integral equation gives rise to a graphical expansion in powers of u. Denoting the bare propagator $\exp[(r - \boldsymbol{k}^2)(t - t')]$ by a straight line, the initial condition by a cross, $h(t)$ by a dash, and $-u/n$ by a dotted line, gives the graphical representation solution shown in figure 2.

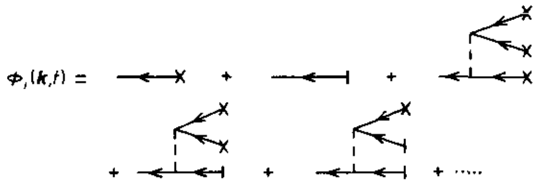

Figure 2. Perturbation expansion for the order-parameter field in powers of the coupling constant u of the time-dependent Landau–Ginzburg equation. A continuous line represents the 'bare' response function $\exp(r - \boldsymbol{k}^2)t - t'$, a broken line the vertex u, a cross the initial condition and a short vertical line the external field h. Arrows indicate the direction of increasing time.

It is useful to define the response function $G(\boldsymbol{k}, t)$ (the 'response to the initial condition'), which is

$$G(\boldsymbol{k}, t) = \left[\frac{\partial \phi_i(\boldsymbol{k}, t)}{\partial \phi_i(\boldsymbol{k}, 0)}\right] \tag{41}$$

and independent of i. The response $G_0(\boldsymbol{k}, t)$ for $h = 0$ and $m_0 = 0$ is written in terms of diagrams in figure 3, where the circle represents the second cumulant Δ of the initial condition, as in (38).

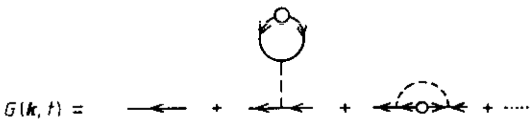

Figure 3. Perturbation expansion for the response function $G(\boldsymbol{k}, t)$. A circle represents the second cumulant Δ of the initial conditions. The other graph elements are explained in the caption to figure 2.

For Gaussian initial conditions one can derive, using either integration by parts or diagrammatic expansion, the relation

$$C_0(\boldsymbol{k}, t) = \Delta G_0(\boldsymbol{k}, t). \tag{42}$$

Therefore

$$G_0(0, t) \propto L(t)^{\lambda} \tag{43}$$

where the subscript 0 again indicates $h = 0$ and $m_0 = 0$.

We want to emphasize that the following results do not require any large-n approximation; they are valid for all n. The price we have to pay is that $G_0(\boldsymbol{k}, t)$, $m(t)$ and λ cannot be calculated explicitly any more, but they can be related to each other.

3.2.1. $h(t) = 0$. We are interested in the average of the order parameter $[\phi_i(\boldsymbol{0}, t)] = \sqrt{N} m(t)$ (by symmetry, only the zero mode has a non-zero average). Figure 4 shows the graphical expansion. A square has been introduced as a new element, which symbolizes the initial magnetization $\sqrt{N} m_0$.

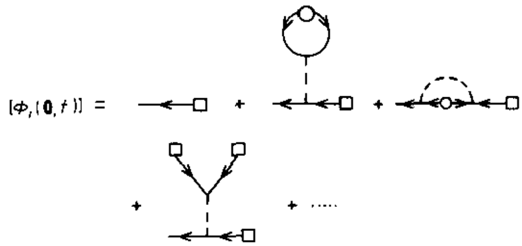

Figure 4. Perturbation expansion for the mean magnetization $m(t)$. A square represents the initial value m_0 of $m(t)$.

If we now only keep terms linear in m_0, we just recover the diagrams of $G_0(\boldsymbol{0}, t)$ with an additional square attached to each diagram, which is

$$[\phi_i(\boldsymbol{0}, t)] = \sqrt{N} m_0 G_0(\boldsymbol{0}, t) + \mathrm{O}(m_0^3)$$

$$m(t) = m_0 G_0(\boldsymbol{0}, t) + \mathrm{O}(m_0^3). \tag{44}$$

(There is no quadratic term in m_0, because $[\phi_i(\boldsymbol{0}, t)]$ is an odd function of m_0.) Hence to first order in m_0, $m(t)$ grows with the same exponent λ as G_0, i.e. $m(t) \sim m_0\, L(t)^\lambda$.

A word of caution is necessary as regards the interpretation of equation (44). It might be thought that it is sufficient to choose m_0 small enough in order to remain in the regime where (44) is valid. This is not true, however, because the higher powers of m_0 also have time-dependent coefficients. For $t = 0$ equation (44) is exact because all coefficients but the linear one vanish, but when $m(t)$ approaches its saturation value of unity, the higher order contributions cannot be neglected any more: the linear approximation breaks down no matter how small m_0 is chosen (the smaller m_0 is, however, the longer the linear approximation is valid). From the fixed-length spin calculation it can be seen that the saturation is approached exponentially as already discussed in [8].

3.2.2. $h(t) \neq 0$. For definiteness we take $m_0 = 0$. For non-zero external field, $m(t)$ is related to the more general response function $G_0(\boldsymbol{k}, t, t_1)$, defined by

$$G_0(\boldsymbol{k}, t, t_1) = \left[\frac{\delta \phi_i(\boldsymbol{k}, t)}{\delta h_i(\boldsymbol{k}, t_1)}\right]_{h \to 0}. \tag{45}$$

This quantity was introduced in [4] as a response to thermal fluctuations. For a general time-dependent external field, (45) implies (for $m_0 = 0$)

$$m(t) = \int_0^t \mathrm{d}t_1 G_0(0, t, t_1)\, h(t_1) + \mathrm{O}(h^3). \tag{46}$$

To determine the form of $G_0(\mathbf{0}, t, t_1)$ we consider the special case $h(t) = m_1 \delta(t - t_1)$, corresponding to an external field pulse at time t_1. Then (46) gives

$$m(t) = m_1 G_0(\mathbf{0}, t, t_1) + \mathrm{O}(m_1^3) \qquad t > t_1. \tag{47}$$

The idea is to write down a scaling form for $G_0(\mathbf{0}, t, t_1)$, based on the idea that the magnetization induced by the pulse should subsequently grow as $L(t)^\lambda$, as discussed in the previous subsection. Furthermore, on scaling grounds we expect the dependence on $L(t)$ to enter through the ratio $L(t)/L(t_1)$. Therefore, if we know the value $G_0(\mathbf{0}, t_1+, t_1)$ of the response function immediately after the pulse (which, from (47), is proportional to the magnetization induced by the pulse) we can write down the scaling form

$$G_0(\mathbf{0}, t, t_1) = G_0(\mathbf{0}, t_1+, t_1)\, f\left(\frac{L(t)}{L(t_1)}\right) \qquad t > t_1 \tag{48}$$

with $f(1) = 1$ and $f(x) \sim x^\lambda$ for large x. However, integrating the Landau–Ginzburg equation

$$\partial\phi/\partial t = (r + \nabla^2)\phi - (u/n)(\phi^2)\phi + m_1\delta(t - t_1)$$

from t_1- to t_1+ gives immediately $m(t_1+) = m_1$, i.e.

$$G_0(\mathbf{0}, t_1+, t_1) = 1. \tag{49}$$

This last result, though exact, is also misleading and cannot be used directly in (48). The reason is that, in the soft-spin Landau–Ginzburg theory, part of the effect of the field pulse is to change the 'length' of the spins, i.e. the system is driven away from the minima of the potential. Such length changes quickly relax away (on a timescale $1/2r$, the relaxation time for 'longitudinal' fluctuations) after the pulse is switched off, leading to a rapid initial *decrease*, to a new quasi-equilibrium value, of the magnetization induced by the pulse, before the eventual slow increase in $m(t)$ begins. The time t_1+ in (48) should be chosen *after* the rapid initial relaxation has occurred.

To clarify this point, it is instructive to consider the fixed-length spin model (6). Taking once more the field to be a pulse, $\boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{m}_1\delta(t - t_1)$, integrating from t_1- to t_1+, and defining $\boldsymbol{m}(t)$ as the volume average of $\boldsymbol{S}(\boldsymbol{x}, t)$, gives

$$m_i(t_1+) = \sum_j (\delta_{ij} - \langle S_i(\boldsymbol{x}, t_1) S_j(\boldsymbol{x}, t_1)\rangle)\, m_{1j}$$

where the angle brackets indicate a volume average. Since the (unit-length) spins are isotropically distributed before the pulse is applied, this becomes $\boldsymbol{m}(t_1) = \{(n - 1)/n\}\boldsymbol{m}_1$, i.e.

$$G_0(\mathbf{0}, t_1+, t_1) = \frac{n-1}{n}. \tag{50}$$

Comparing (50) with (49) we see that, for soft spins, a fraction $1/n$ of the induced magnetization is associated with 'spin stretching', and the remainder with 'spin rotation'. The difference between (49) and (50) will become qualitatively important for $n = 1$.

For $n > 1$, the above discussion leads to the scaling form

$$G_0(0,t,t_1) = \frac{n-1}{n} f\left(\frac{L(t)}{L(t_1)}\right) \qquad t > t_1 \tag{51}$$

with $f(1) = 1$ and $f(x) \sim x^\lambda$ for $x \gg 1$. Using this in (46), with $L(t) \sim t^{1/2}$ and $h(t_1) = h =$ constant, and changing variables to $y = t_1/t$, gives

$$m(t) = \frac{n-1}{n} ht \int_{t_0/t}^{1} \mathrm{d}y f\left(\frac{1}{\sqrt{y}}\right) \tag{52}$$

where we have once more introduced a lower cut-off t_0 on the time integral to represent the short-time limit of the scaling regime. Since $f(x) \sim x^\lambda$ for large x, the integral will converge at the lower limit if $\lambda < 2$, to give $m(t) \sim ht$ plus subdominant terms. For $\lambda > 2$, on the other hand, the integral is dominated by the vicinity of the lower limit, and one obtains $m(t) \sim ht_0(t/t_0)^{\lambda/2}$ plus subdominant terms. These results generalize the large-n results obtained in section 2. Note that both terms (in t and $t^{\lambda/2}$) are present in general, but which one of them dominates for large t depends on the value of λ. In practice, one should include both terms in the fit, as is evident from a study of figure 1: there is no dramatic change of behaviour at $d = 4$ (which corresponds to $\lambda = 2$ for $n = \infty$), but rather a smooth crossover as a function of d.

The case $n = 1$, corresponding to a scalar order parameter, has to be considered separately. The domain state consists of a set of well-defined domains, in which the local order parameter is close to one of the two equilibrium values $\pm(r/u)^{1/2}$, separated by narrow walls of width $w \sim 1/\sqrt{r}$. When a magnetic field pulse is applied, the induced magnetization inside the domains quickly relaxes away when the field is switched off. The important effect of the field occurs at the domain walls, where it leads to a shift in the wall positions so as to increase the global magnetization. Shortly after the field has been switched off, the residual magnetization is proportional to the volume fraction occupied by the walls at time t_1, which is of order $w/L(t_1)$. The analogue of equation (51) is

$$G_0(0,t,t_1) = \frac{w}{L(t_1)} f\left(\frac{L(t)}{L(t_1)}\right) \qquad t > t_1. \tag{53}$$

Putting this in (46) (with $h(t) = h =$ constant) gives, up to constants,

$$m(t) = h\sqrt{t} \int_{t_0/t}^{1} (\mathrm{d}y/\sqrt{y})\, f(1/\sqrt{y}). \tag{54}$$

This time the integral converges at the lower limit if $\lambda < 1$, to give $m(t) \sim h\sqrt{t}$ plus subdominant terms. For $\lambda > 1$ the vicinity of the lower limit dominates and $m(t) \sim ht_0(t/t_0)^{\lambda/2}$ is obtained.

To conclude this section we note that it is simple to incorporate both a weak magnetic field h and a small initial magnetization m_0, since the effects are additive to leading order in m_0 and h:

$$m(t) = h(at^x + bt^{\lambda/2}) + cm_0 t^{\lambda/2} \tag{55}$$

where a, b and c are constants and $x = 1$ ($\frac{1}{2}$) for a vector (scalar) order parameter. Note that the arguments given above imply (i) $a > 0, b < 0$ for $\lambda < \lambda_c$, (ii) $a < 0, b > 0$ for $\lambda > \lambda_c$, (iii) $c > 0$ always, where $\lambda_c = 2$ for $n > 1$ and $\lambda_c = 1$ for $n = 1$. These results agree with the explicit results obtained for large n. For a system quenched from a high-temperature equilibrium state in a weak magnetic field, m_0 will itself be linear in h.

4. Summary

In section 2 the large-n limit was used to investigate the phase-ordering dynamics of a system quenched into the ordered phase under conditions corresponding to an off-critical quench, i.e. either the initial magnetization m_0 or the external field h (or both) is non-zero. One of the central results is that for $h = 0$ the time evolution of the magnetization $m(t)$ is governed by the same exponent λ that controls dynamic correlations during phase ordering, i.e. $m(t) \sim m_0 L(t)^\lambda$, where $L(t) \sim t^{1/2}$ is the characteristic length scale. This result is valid as long as $m(t) \ll 1$, i.e. the magnetization is small compared to its saturation value. (At general temperatures below T_c one would require $m(t)$ to be much smaller than the *equilibrium* magnetization, which in general will be smaller than the *saturation* value, due to thermal fluctuations). In section 3 the result $m(t) \sim m_0 L(t)^\lambda$ was shown to follow quite generally from the structure of the diagrammatic perturbation expansion for $m(t)$.

The second main result is a prediction for the behaviour of $m(t)$ when a weak external field is present. From the large-n calculation, and general scaling considerations, we expect a result of the form (55), containing two competing power laws. As well as an explicit term in $t^{\lambda/2}$ multiplying the initial magnetization, there is another such term, linear in h, which represents the magnetization induced by the magnetic field on short times. The remaining term, linear in both h and t (or $\sqrt{t}$ for $n = 1$), is due to the continued effect of the field at late times.

From an experimental viewpoint, $m(t)$ (or, more generally, the mean order parameter) is measureable in principle from the Bragg peak intensity (proportional to $m(t)^2$) in a scattering experiment. In the presence of a magnetic field (or, more generally, the field conjugate to the order parameter), there are two competing power laws. It is suggested that both of these be included in fits of the data to (55). In conclusion we hope that the present paper will stimulate experimental attempts to measure the non-trivial exponent λ.

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Appendix. Abel's equation

We want to cast equation (28) into a standard differential equation, which will be Abel's equation. For convenience, let us choose $m_0 = 0$, and for brevity define

$$s(t) = \left(\frac{1}{N}\sum_p e^{-2p^2t}\right)^{1/2} . \tag{A1}$$

Equation (28) then becomes

$$\left(\frac{\dot{f}(t)}{h(t)}\right)^2 = s(t)^2 + f(t)^2 \tag{A2}$$

with initial condition $f(0) = 0$. For $h(t) \geqslant 0$ and $h(t) = 0$ only for isolated points, we can redefine time by a one-to-one mapping:

$$\tau = \int_0^t \mathrm{d}t' h(t') \tag{A3}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = h(t). \tag{A4}$$

The left-hand side of (A2) involves the derivative of f with respect to τ:

$$\frac{\mathrm{d}f(t(\tau))}{\mathrm{d}\tau} = \frac{\mathrm{d}f(t)}{\mathrm{d}t}\frac{1}{h(t)}. \tag{A5}$$

We introduce the functions

$$f'(\tau) = f(t(\tau)) \tag{A6}$$

$$s'(\tau) = s(t(\tau)). \tag{A7}$$

In this notation (A2) reads

$$\left(\frac{\mathrm{d}f'(\tau)}{\mathrm{d}\tau}\right)^2 = s'(\tau)^2 + f'(\tau)^2 \tag{A8}$$

with the initial condition $f'(0) = 0$.

The next substitution to be applied is the introduction of $u(\tau)$:

$$f'(\tau) = s'(\tau)\sinh u(\tau). \tag{A9}$$

With the trigonometric identity $1 + \sinh^2 u(\tau) = \cosh^2 u(\tau)$, (A8) then becomes

$$\dot{u}(\tau) = 1 - \frac{\dot{s}'(\tau)}{s'(\tau)}\tanh u(\tau). \tag{A10}$$

The dot is now the derivative with respect to τ. The initial condition transforms to $u(0) = 0$. Equation (A10) itself could be used as a starting point for approximations. With one last substitution

$$w(\tau) = \tanh u(\tau) \tag{A11}$$

we arrive at Abel's equation:

$$\dot{w}(\tau) = (1 - w(\tau)^2)\left(1 - \frac{\dot{s}'(\tau)}{s'(\tau)}w(\tau)\right) \tag{A12}$$

and $w(0) = 0$. For further discussion of this nonlinear first-order differential equation see [10].

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